*Journal of Applied Mechanics and Technical Physics, Vol. 50, No. 2, pp. 270–277, 2009*

# KINEMATIC PROBLEM OF AN IDEAL INCOMPRESSIBLE FLUID FLOW AROUND AN ARBITRARILY MOVING AIRFOIL

**A. I. Zobnin**

UDC 532.5: 533.6

*An unsteady kinematic problem for arbitrary two-dimensional motion of an airfoil in an ideal incompressible fluid with formation of one and two vortex wakes is solved. The problem is solved by the method of conformal mapping of the flow domain onto a circle exterior; solution singularities in the vicinity of a sharp edge are analyzed, and the initial asymptotics of the solution is taken into account. The calculated results are found to be in good agreement with available experimental data on visualization of the flow pattern. The necessity of correct modeling of the initial stage of vortex-wake formation is demonstrated. A regular flow pattern is found to form after three and more periods of oscillations.*

**Key words:** *airfoil, two-dimensional problem, ideal fluid, vortex wake.*

The method of conformal mapping is widely used in solving problems of an ideal incompressible fluid flow around an airfoil. In the general case of arbitrary motion of the airfoil, the flow pattern can be constructed by the method used in [1]. As was noted in [1], however, cumbersome transformations are needed even for the simplest laws of airfoil motion. The simplest case is the translational motion of the airfoil [2–6]. An idea simplifying obtaining effective expressions for the solution with allowance for rotation was also proposed in [1]. Even after this simplification, however, a particular numerical calculation is not very simple.

The problem of the flow around airfoils with formation of vortex wakes modeled by tangential discontinuities of the velocity field is solved in the present paper by the method of conformal mapping. The approach proposed here, in contrast to that considered in [1], is more convenient for numerical implementation of solving the problem, which is particularly important if the airfoil motion is not rigorously translational. The law of motion is determined by displacement of the center fixed in a certain coordinate system fitted to the airfoil and by rotation of the airfoil with an unchanged shape around a chosen point.

**1. Formulation of the Problem.** We consider two-dimensional unsteady motion of an ideal incompressible fluid around an airfoil under the condition that there are no mass forces and initial vorticity (in this case, the Thompson theorem is valid). The airfoil moves in one plane from the state at rest by an arbitrary law. The flow around the airfoil involves the formation of vortex wakes in the form of velocity discontinuity lines, which are shed from sharp and smooth edges. Outside the airfoil contour and discontinuity lines, the flow is assumed to be potential, and the fluid particle velocities are assumed to be finite.

Let us introduce a motionless rectangular coordinate system $O_1x_1y_1$ in which the fluid is at rest at infinity. Let the airfoil points move with a velocity $-\boldsymbol{U}_\infty(x_1, y_1, t)$ (Fig. 1).

To determine the fluid particle velocity $\boldsymbol{v}(x_1, y_1, t)$, we formulate the following initial-boundary problem. The airfoil contour is denoted by $L_0$, and the wake contours (lines of tangential discontinuities of velocity) are

Omsk Department of the Sobolev Institute of Mathematics, Siberian Division, Russian Academy of Sciences, Omsk 644099; alzobnin@mail.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 50, No. 2, pp. 120–128, March–April, 2009. Original article submitted October 29, 2007; revision submitted February 20, 2008.

 

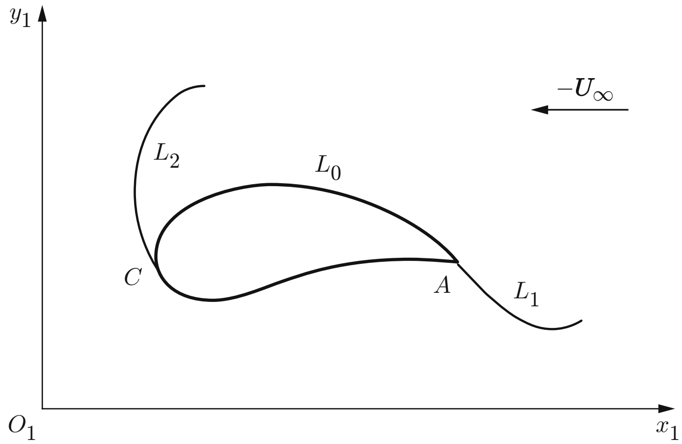


Fig. 1. Layout of the problem.

denoted by $L_1$ and $L_2$. The fluid motion outside the contour $L = L_0 + L_1 + L_2$ satisfies the Euler equation

$$\frac{d\boldsymbol{v}}{dt} = -\frac{1}{\rho}\nabla p$$

($\rho$ is the density and $p$ is the pressure) and also the equation of continuity and the condition of potentiality:

$$\operatorname{div}\boldsymbol{v} = 0, \qquad \operatorname{rot}\boldsymbol{v} = 0.$$

The points of the contour $L_0$ are subjected to the no-penetration condition

$$(\boldsymbol{v} + \boldsymbol{U}_\infty)\cdot\boldsymbol{n}_0 = 0 \quad \text{for} \quad (x_1, y_1) \in L_0(t),$$

the vortex wakes are subjected to the condition of the absence of pressure differences

$$[p] = 0 \quad \text{for} \quad (x_1, y_1) \in L_1(t), \quad (x_1, y_1) \in L_2(t)$$

and continuity of the normal-to-wake component of fluid velocity

$$\boldsymbol{v}\cdot\boldsymbol{n} = v_n \quad \text{for} \quad (x_1, y_1) \in L_1(t), \quad (x_1, y_1) \in L_2(t),$$

and the condition of decay of perturbed velocities is imposed at infinity:

$$\lim \boldsymbol{v} = 0 \quad \text{as} \quad (x_1, y_1) \to \infty.$$

In addition, we require the Chaplygin–Joukowski postulate to be satisfied at the points of vortex-wake shedding:

$$|\boldsymbol{v}| < \infty \quad \text{as} \quad (x_1, y_1) \to (x_{1A}, y_{1A}), \quad (x_1, y_1) \to (x_{1C}, y_{1C}).$$

The following conditions are used as the initial data:

$$\boldsymbol{v}\Big|_{t=0} = 0, \qquad \boldsymbol{U}_\infty\Big|_{t=0} = 0, \qquad L\Big|_{t=0} = L_0\Big|_{t=0}.$$

In the formulas cited above, $\boldsymbol{n}_0$ and $\boldsymbol{n}$ are the unit vectors to $L_0$ and $L_1 + L_2$, respectively, $(x_{1A}, y_{1A})$ and $(x_{1C}, y_{1C})$ are the coordinates of the vortex shedding points, and $v_n$ is the velocity of motion of the vortex points in the normal direction.

The initial-boundary problem formulated above is divided into two parts: 1) Riemann–Hilbert boundary-value problem on determining the velocity field at each instant of time [2]; 2) Cauchy problem for the integrodifferential equation of motion of the vortex-wake points.

1.1. *Formulation of the Riemann–Hilbert Problem.* Let us consider the airfoil and the vortex wakes at a certain fixed instant of time $t > 0$ in a complex plane $z_1 = x_1 + iy_1$ (see Fig. 1). Let the wake contours $L_1$ and $L_2$ with the tracking direction from the shedding point to the free end and the intensities of these wakes be known. Then, for the complex velocity of fluid particles $\bar{v}(t, z_1)$, we can formulate a boundary-value problem on constructing a function $\bar{v}(t, z_1)$ outside the contours $L_0$, $L_1$, and $L_2$, which satisfies the following conditions:

1) $\bar{v}(t, z_1)$ is an analytical function of the complex variable $z_1$;

2) the fluid does not penetrate through the contour $L_0$ (no-penetration condition):

$$\operatorname{Re}[n_0(\zeta)\,\bar{v}(\zeta)] = -\operatorname{Re}[n_0(\zeta)\,\bar{U}_\infty], \qquad \zeta \in L_0$$

[$n_0(\zeta)$ is a complex form of recording of the unit vector normal to $L_0$ and $-\bar{U}_\infty$ is the complex velocity of the airfoil points];

3) the complex velocity $\bar{v}(t, z_1)$ has a prescribed jump $\gamma$ on the lines $L_1$ and $L_2$:

$$\bar{v}^+(\zeta) - \bar{v}^-(\zeta) = \gamma(\zeta), \qquad \zeta \in L_1 + L_2$$

[$\bar{v}^+$ and $\bar{v}^-$ are the limiting values of $\bar{v}(t, z_1)$ when the contour $L_1$ or $L_2$ is approached from the right and from the left, respectively];

4) the complex velocity $\bar{v}(t, z_1)$ decays at infinity:

$$\bar{v}(t, z_1) \to 0 \quad \text{as} \quad |z_1| \to \infty;$$

5) the function $\bar{v}(t, z_1)$ is finite everywhere in the domain of its definition (and the Chaplygin–Joukowski condition is also satisfied):

$$|\bar{v}(t, z_1)| < \infty;$$

6) circulation of velocity over an arbitrary closed fluid contour enclosing the airfoil and the vortex wakes equals zero (the Thompson theorem is satisfied):

$$\Gamma_0 = 0$$

($\Gamma_0$ is the circulation of velocity over the contour $L = L_0 + L_1 + L_2$).

Problem 1–6 is the Riemann–Hilbert problem and admits a unique solution if the contour $L = L_0 + L_1 + L_2$ and the velocity jump $\gamma$ are given [7].

1.2. *Cauchy Problem for the Integrodifferential Equation of Motion of the Vortex-Wake Points.* With allowance for the solution $\bar{v}(t, z_1)$ of the Riemann–Hilbert problem 1–6 and the first integral of the system of equations of the initial problem (Cauchy–Lagrange integral)

$$p = p_\infty - \rho\Big[\frac{\partial}{\partial t}\operatorname{Re}\Big(\int_{z_0}^{z_1}\bar{v}(z)\,dz\Big) + \frac{1}{2}\,v\bar{v}\Big],$$

the condition of the absence of the pressure jump on the wake contours implies that the circulation $\Gamma$ at the vortex-wake point, calculated from the free end of the contour, remains unchanged at the points moving with the velocity $\bar{v}^0 = (\bar{v}^+ + \bar{v}^-)/2$ ($p_\infty$ is the pressure at infinity).

If the vortex-wake contour ($L_1$ or $L_2$) is defined in parametric form $\zeta = \zeta(t, \Gamma)$, the motion of its points is determined by the following Cauchy problem for the integrodifferential equation:

$$\frac{\partial\bar{\zeta}}{\partial t}(t, \Gamma) = \bar{v}^0(t, \zeta(t, \Gamma)), \qquad \zeta(t, \Gamma)\Big|_{t=0, \Gamma=0} = \zeta^*(t)\Big|_{t=0}.$$

Here $\zeta^*$ is the complex coordinate of the shedding point ($L_1$ or $L_2$). The solution of the Cauchy problem formulated above has to be constructed in the domain $t \geq 0$, $0 \leq \Gamma \leq \Gamma^*(t)$, where $\Gamma^*(t)$ is the value of the circulation $\Gamma$ at the wake shedding point. The solution $\zeta(t, \Gamma)$ is assumed to ensure continuity of $\bar{v}$ in terms of $z_1$ in the entire flow domain outside $L$, continuity of $\bar{v}$ in terms of $t$ up to the initial time, and finiteness of the velocity field (including satisfaction of the Chaplygin–Joukowski condition). In addition, we require that the second derivative of the function $\Gamma(\zeta)$ for each value $t > 0$ belong to the class $H^*$ in the vicinity of the free end of the wake and to the class $H$ on the remaining part of the wake [7].

**2. Solution of the Problem.** Let the motion of a certain center $F$ of airfoil rotation $z_{1F} = z_{1F}(t)$ be defined at each time instant in a complex plane $z_1 = x_1 + iy_1$, related to the inertial Cartesian coordinate system $O_1x_1y_1$, with the fluid being at rest at infinity. Let the angle $\varphi(t)$ of airfoil rotation, counted from the real axis, be also known (Fig. 2).

Conformal mapping of the exterior of the contour onto the exterior of a circle lying in another complex plane is assumed to be known. A formula for the unsteady flow potential with rotational motion of the airfoil was

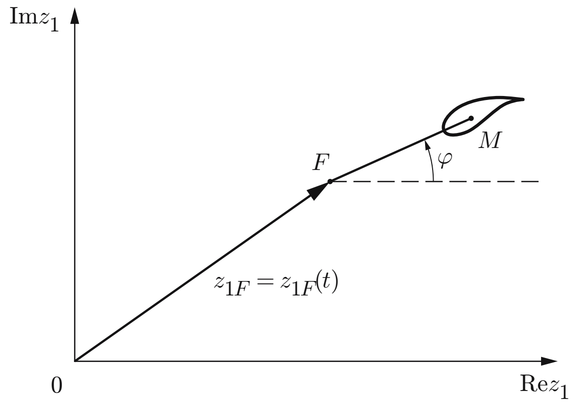


Fig. 2. Scheme of airfoil motion.

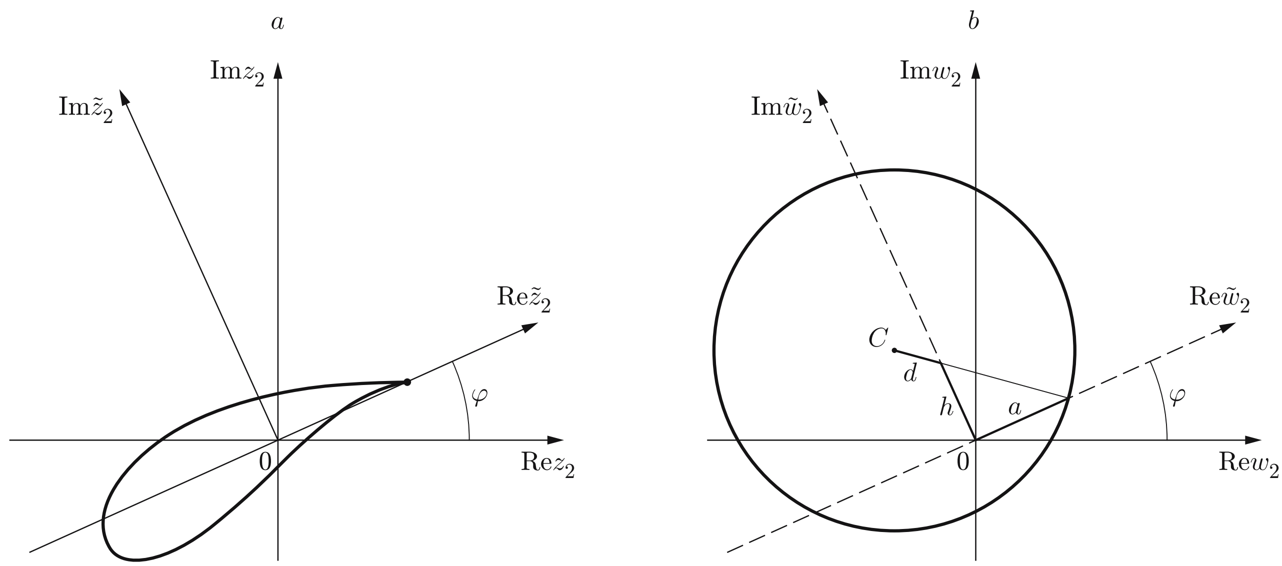


Fig. 3. Complex planes $z_2$ (a) and $w_2$ (b).

derived in [1] under the assumption that the mapping is a rational function. In the present work, we assume that the contour is the Kármán–Trefftz airfoil with parameters $a$, $h$, $d$, and $\delta$ determining the chord length, curvature, thickness, and angle of the sharp edge, respectively. This means that we can introduce a complex plane $\tilde{z}_2$ at the time $t$ (Fig. 3), such that the mapping

$$\frac{\tilde{z}_2 - \delta_1 a}{\tilde{z}_2 + \delta_1 a} = \Big(\frac{\tilde{w}_2 - a}{\tilde{w}_2 + a}\Big)^{\delta_1}, \qquad \delta_1 = 2 - \delta \tag{1}$$

transforms the region external to the airfoil in this plane to the exterior of a circle in the plane $\tilde{w}_2$. The axes $\mathrm{Re}\, \tilde{z}_2$ and $\mathrm{Im}\, \tilde{z}_2$ form an angle $\varphi$ with the axes $\mathrm{Re}\, \tilde{z}_1$ and $\mathrm{Im}\, \tilde{z}_1$. The point $M$ in Fig. 2 corresponds to the point $\tilde{z}_2 = 0$. The position of this point with respect to the airfoil is uniquely determined for each Kármán–Trefftz airfoil. After the substitution $\tilde{z}_2 = z_2 \,\mathrm{e}^{-i\varphi}$ and $\tilde{w}_2 = w_2 \,\mathrm{e}^{-i\varphi}$, transformation (1) becomes the mapping

$$\frac{z_2 - \delta_1 a \,\mathrm{e}^{i\varphi}}{z_2 + \delta_1 a \,\mathrm{e}^{i\varphi}} = \Big(\frac{w_2 - a \,\mathrm{e}^{i\varphi}}{w_2 + a \,\mathrm{e}^{i\varphi}}\Big)^{\delta_1}, \qquad \delta_1 = 2 - \delta, \tag{2}$$

which transforms the airfoil rotating by the angle $\varphi$ into a circumference rotating simultaneously with the airfoil around the origin (see Fig. 3).

It is convenient to construct the solution of the Riemann–Hilbert boundary-value problem in the plane $w$ whose axes are parallel to the axes $\mathrm{Re}\, w_2$ and $\mathrm{Im}\, w_2$, and the origin is located at the point $w_{2C}(t)$, which is the center of the circumference $C$: $w = w_2 - w_{2C}(t)$. The domain of solution construction is canonical, which allows us to use inversion of the wakes with respect to the circumference to obtain the explicit form of the function. Moreover, the solution remains potential; therefore, the potential as a function of the spatial variable $w$ can be found as the sum of the potentials, one of the latter being induced by vortex wakes.

If the wake shedding point is a regular point of the mapping (in the case of a smooth edge), then the Chaplygin–Joukowski condition 5 (as well as conditions 1 and 2) has the same form in the plane $w$ as that in the initial plane $z_1$. In the case of a sharp edge, the mapping conformity is violated, the type of the singularity depends on the type of the edge, and the mapping in the plane $w$ should have the zero of the corresponding order at this point. The condition of decay at infinity (condition 4) is transformed with allowance for the results of the analysis of the asymptotic behavior of the derivative of mapping (2) at infinity. As $\partial w_2/\partial z_2 = 1$ with $z_2 \to \infty$, the condition at infinity remains unchanged as we pass from the plane $z_2$ to the plane $w_2$:

$$-U_0 = -\frac{dz_{1F}(t)}{dt} + iz_* \,\mathrm{e}^{i\varphi}\, \varphi'.$$

Here $z_*$ is the complex coordinate $z_2$ of the center of airfoil rotation at the time $t = 0$; the prime denotes differentiation with respect to time. As the center $C$ of the plane $w$ rotates around the point $w_2 = 0$ with a velocity $dw_{2C}(t)/dt = i\varphi' \,\mathrm{e}^{i\varphi}\, w_{2C}(0)$, the condition at infinity in this plane (condition 4) is finally written as

$$-U_\infty = -\frac{dz_{1F}(t)}{dt} + i\varphi' \,\mathrm{e}^{i\varphi}(z_* - w_{2C}(0)).$$

If the vortex wakes (their shapes and intensities) are defined at the time $t$, then the solution of the Riemann–Hilbert problem can be presented in the following form with allowance for the above-given considerations [7]:

$$\frac{\partial\hat{\Phi}}{\partial w} = (-\bar{U}_\infty) - \frac{R^2(-U_\infty)}{w^2} + \frac{1}{2\pi i}\sum_{j=1,2}\int_0^{\Gamma_j^*}\Big(\frac{1}{\zeta_j - w} - \frac{1}{R^2/\zeta_j - w}\Big)\, d\Gamma_j. \tag{3}$$

Here $\hat{\Phi}$ is the potential in the plane $w$, $R$ is the circumference radius, $\zeta_j$ is the current point of the $j$th wake, and $\Gamma_j$ and $\Gamma_j^*$ are the total intensities of the $j$th wake at the point $\zeta_j$ and in the entire $j$th wake, respectively.

Solution (3) constructed for the canonical domain allows us to obtain the solution of the initial kinematic problem. For this purpose, we need to take into account the relation between the potential $\hat{\Phi}(w)$ of the plane $w$ and the potential of absolute motion $\Phi_1(z_1)$. Indeed, as $\partial z_2/\partial t = \partial z_1/\partial t - U_0$ and $\partial\Phi_1/\partial z_2 = \partial\Phi/\partial z_2 + \bar{U}_0$, then the equation for fluid particles

$$\frac{\partial\bar{z}_2}{\partial t} = \frac{\partial\Phi_1}{\partial z_1} \tag{4}$$

can be written in the form $\partial\bar{z}_2/\partial t = \partial\Phi/\partial z_2$. The potential $\Phi$ in the plane $z_2$ can be expressed via the potential in the domain $w_2$ with the help of the mapping derivative: $\partial\Phi/\partial z_2 = (\partial\Phi/\partial w_2)(\partial w_2/\partial z_2)$. In turn, the solution in the plane $w_2$ with the potential $\Phi(w_2)$ is related to the potential $\hat{\Phi}(w_2)$ by the formula $\partial\Phi/\partial w_2 = \partial\hat{\Phi}/\partial w_2 + dw_{2C}/dt$. In the latter equality, $\hat{\Phi}(w_2)$ coincides with the potential $\hat{\Phi}(w)$ from Eq. (3), which is expressed via the variable $w_2$. Thus, the sought relation between the potentials can be written in the form of the equality

$$\frac{\partial\Phi_1}{\partial z_1} = \bar{U}_0 + \frac{\partial w_2}{\partial z_2}\Big(\frac{\partial\hat{\Phi}}{\partial w} + \frac{dw_{2C}}{dt}\Big). \tag{5}$$

Relations (3) and (5) determine the solution of the boundary-value problem for the function of the complex velocity in the initial variables. Solution (5) is fairly simple and is used in the numerical solution of the Cauchy problem for determining the motion of the vortex-wake points.

The equation of the problem that describes the evolution of vortex wakes follows from Eqs. (3)–(5) and from the dynamic condition on the vortex sheet:

$$\frac{\partial\bar{z}_1}{\partial t} = \bar{U}_0 + \frac{\partial w_2}{\partial z_2}\Big(\frac{\partial\hat{\Phi}_0}{\partial w} + \frac{dw_{2C}}{dt}\Big). \tag{6}$$

Here $\partial\hat{\Phi}_0/\partial w$ is the integral in the sense of the principal value. The Cauchy problem for the functions $z_1(t, \Gamma_j)$, where $j = 1, 2$, is obtained by adding the initial data about the quiescent state and the data corresponding to the absence of vortex wakes to Eq. (6). In this procedure, only solutions satisfying the Chaplygin–Joukowski condition should be considered. This means, in particular, that the wake and the airfoil should have a common tangential line at the shedding point.

The problem formulated here is nonlinear, because the boundary of the solution-construction domain $t \geq 0$, $0 \leq \Gamma_j \leq \Gamma_j^*$ $(j = 1, 2)$ for each wake is not known in advance and has to be determined along with construction of the sought functions. Another specific feature of the problem is the solution instability in the Hadamard sense, which requires particular care in modeling the initial stage of vortex-wake formation. Without considering the details of this issue, we should note that the approach developed in [8–10] for different types of airfoil edges for the case of translational motion can be extended to the case of arbitrary two-dimensional motion of the airfoil.

The character of relations that describe vorticity shedding is extremely important for both theoretical aspects and practice. Numerical research (see, e.g., [2]) show that the contribution of vortex wakes plays the governing role in formation of forces acting on the airfoil. In turn, the influence of the wake portion closest to the shedding point, in particular, its shape and intensity, on the parameters of the next portion of vorticity being shed is also quite significant [2]. The convergence of the integrals in the right side of Eq. (6) at singular points of junction between the wakes and the airfoil follows from the character of variation of the vortex-wake intensity near the edge [10] and from the equality of the vortex-wake curvature at the point of its shedding to the airfoil curvature.

The relations for vorticity shedding with the translational law of motion were given in [2]. In the case considered here, the relations are similar with the only exception that the wake shedding points in the plane $w$ are additionally displaced over the circumference in accordance with the specified law of rotation $\varphi(t)$. For instance, the Chaplygin–Joukowski postulate for the corner-shaped edge acquires the form

$$-2U_\infty \sin(\theta + \theta_1 - \varphi) - \varphi' a \cos\theta_1 + \frac{1}{2\pi} \int\limits_{K_1, K_2} \Big( \frac{2\sigma_1}{\sigma_1^2 + \sigma_2^2} + R^{-1} \Big)\, d\Gamma = 0.$$

Here $\theta = \arg(-U_\infty)$ and $\theta_1 = \arctan(h/a)$; $K_1$ and $K_2$ are the wake contours in the domain of conformal mapping; $\sigma_1$ and $\sigma_2$ are the abscissa and ordinate of the Cartesian coordinate system with the origin at the point of junction between the circumference and the image of the wake shed from the corner-shaped edge.

With allowance for the comments given above, the Cauchy problem for the evolution of vortex wakes can be solved numerically with the use of various methods of interpolation of the discontinuity lines of the velocity field, such as piecewise functions, point vortices, etc. Feasibility of wake discretization with constant intensity of each portion of discretization in time is confirmed by conservation of the total vorticity of the wake element in the course of its motion.

Solving the problem for Eq. (6), the forces acting on the airfoil can be found by integrating the pressure determined by the Cauchy–Lagrange integral over the airfoil contour:

$$-\frac{p}{\rho} = \mathrm{Re}\Big(\frac{\partial \Phi_1}{\partial t}\Big|_{z_1 = \mathrm{const}}\Big) + \frac{1}{2}\Big|\frac{\partial \Phi_1}{\partial z_1}\Big|^2 + F(t).$$

Here $F(t)$ is an arbitrary function of time and $\rho$ is the density. The complex potential of absolute motion $\Phi_1(z_1)$ is differentiated in the absolute coordinate system related to the plane $z_1$. For numerical implementation, however, it seems reasonable to pass to spatial variables of the conformal mapping domain in the formula for pressure and to differentiation in a non-rotating coordinate system fitted to the circumference center. By virtue of the invariance of the derivative in the particle with respect to coordinate system transformation in passing from the variables $z_1$ and $t$ to the variables $w$ and $t$, the equation for pressure is transformed to

$$-\frac{P}{\rho} = \mathrm{Re}\Big(\frac{\partial \Phi_1}{\partial t}\Big|_{w = \mathrm{const}} + \frac{\partial w}{\partial t}\frac{\partial \Phi_1}{\partial w}\Big) - \frac{1}{2}\Big|\frac{\partial \Phi_1}{\partial w}\Big|^2 \Big|\frac{\partial w_2}{\partial z_2}\Big|^2. \tag{7}$$

In this form, the Cauchy–Lagrange integral can be used for calculating hydrodynamic loads. The expressions for $\partial\Phi_1/\partial w$ and $\partial\Phi_1/\partial t$ in the right side of Eq. (7) are obtained by direct differentiation of the potential $\Phi_1$ found by integration of Eq. (5) with respect to the variable $z_1$. The quantity $\partial w/\partial t$ can be described by the presentation

$$\frac{\partial w}{\partial t} = wi\varphi' + \frac{\partial w_2}{\partial z_2}\Big(\Big(\overline{\frac{\partial \Phi}{\partial w}}\Big)\Big(\overline{\frac{\partial w_2}{\partial z_2}}\Big) - z_2 i\varphi'\Big), \tag{8}$$

which follows from the relation of the solution in the physical plane and in the plane of conformal mapping

$$\frac{\partial w_2}{\partial t} - w_2 i\varphi' = \frac{\partial w_2}{\partial z_2}\Big(\frac{\partial z_2}{\partial t} - z_2 i\varphi'\Big),$$

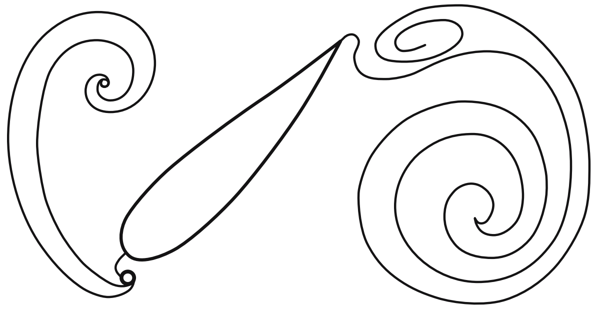

Fig. 4. Vortex wakes in the case of purely rotational oscillations of the airfoil.

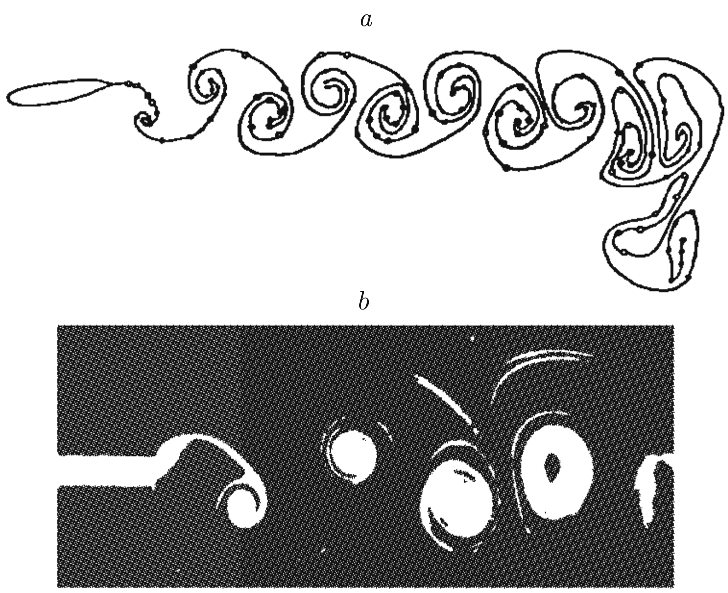

Fig. 5. Vortex streets in the case of rotational oscillations of the airfoil: (a) calculated results; (b) flow visualization.

and also from the equation of motion of the wake $\partial z_2/\partial t = (\overline{\partial\Phi/\partial w})(\overline{\partial w_2/\partial z_2})$ and from the equalities $w_2 = w + e^{i\varphi}\, w_{2C}(0)$ and $\partial w_2/\partial t = \partial w/\partial t + i\varphi'\, e^{i\varphi}\, w_{2C}(0)$. If we choose a point $w$ on the wake in Eq. (8): $w = \zeta$ (which leads to the emergence of integrals in the sense of the principal value), then this formula can be used to find the vortex-point velocities $\partial\zeta/\partial t$, which appear in the expression for $\partial\Phi_1/\partial t$. It should be noted that the derivative $\partial\Phi/\partial w$ in the right side of Eq. (8) coincides with the derivative $\partial\hat{\Phi}/\partial w$ in Eq. (3) with accuracy to the term $dw_{2C}/dt$.

Thus, the Cauchy–Lagrange integral (7) together with appropriate presentations of the quantities involved yields the solution of the problem on determining hydrodynamic reactions on the airfoil. Though the resultant expression for pressure is cumbersome, but it is simple and convenient for numerical calculations. Finally, it should be noted that the proposed method of application of the method of conformal mapping to the problem of the flow around an arbitrarily moving airfoil is an effective basis for the development of numerical algorithms and numerical research. Calculations for hydrodynamic loads were not performed in the present study.

**3. Numerical Solution of the Problem.** A kinematic problem of an ideal incompressible fluid flow around an airfoil with formation of one or two vortex wakes was solved numerically. The calculation scheme as a whole is similar to that used in [2]. The changes in the calculation scheme are caused by extension of the solution to the case of arbitrary motion of the airfoil, as was noted in Secs. 1 and 2. The results calculated for one vortex wake are compared with experimental data on flow visualization [11, 12].

Figure 4 shows an airfoil that performs purely rotational oscillations around the point $z_1 = 0$ with an amplitude $\varphi_0 = \pi/4$ in the absence of the incoming flow.

Formation of vortex wakes on the smooth and corner-shaped edges occurs in a similar manner; as in the case of purely translational oscillations, however, there are some differences [2]. Figure 5 shows the results of calculations and the photograph of the vortex wake in the case of rotational oscillations around a point located at a distance of 1/4 of the airfoil chord $b$ from the leading edge [the amplitude of oscillations is $\varphi_0 = 10°$, the free-stream velocity $V_\infty$ is 0.86 of the chord during the period $T$, the rigging angle (mean position of the airfoil) is $\varphi_{00} = 0$, and the initial phase of oscillations is $\mu = 0$]. In the calculations, we used $t = 6T$. It is seen that a regular vortex street is not formed during the first two periods of oscillations; this happens after three and more periods. The street segment corresponding to $t < 2T$ displays clearly expressed wakes owing to the choice of the initial phase $\mu$ in the form of a lateral pulse to the street direction, which rapidly disappear.

The results of the comparison performed testify that the calculated data are reliable. In particular, in the calculations with the values of parameters used in the experiments, the vortex street does form; it does not become mixed or destroyed, as it happens, for instance, at $t < 2T$. In addition, the calculated pattern of the vortex blobs is consistent with that observed in the experiment.

The results of the numerical study performed and comparisons with visualization data allow us to conclude that the method developed for solving the kinematic problem of the flow around an airfoil and the computational algorithm generated on the basis of this method are applicable for various shapes and laws of motion of airfoils for high amplitudes, angles of attack, and Strouchal numbers.

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